ICME Survey Team: Geometry Hamburg, 2016

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Identifying 7 threads

- developments in the use of theories;
- spatial reasoning;
- the use and role of diagrams and gestures;
- the role of technologies;
- the teaching and learning of definitions;
- the teaching and learning of the proving process;
- moving beyond traditional Euclidean approaches.

Identifying relevant research

- Focus on past 8 years of research
  - Past Handbook chapters
  - International conferences (mainly PME)
  - Peer-reviewed journal articles (ESM, FLM, MTL, RME, JRME, IJMTL, CJSMTE, TKL, JMB, JMTE, IJMEST)
- Writing up summaries for each thread
  - Need to provide context, which requires citing some earlier work
  - Attempt to highlight main results/questions and connections within the research

1. Developments and trends in the use of theories

Led by Keith Jones and Allen Leung

1.1 Different theories on the teaching and learning of geometry

- Theories specifically about geometry education
  - the *van Hiele model* (van Hiele 1986)
  - the theory of *figural concepts* (Fischbein 1993)
  - the theory of *figural apprehension* (Duval 1998)
  - the theory of *geometric work* (c.f., Kuzniak 2014)
- Theories being applied to geometry education
  - *prototype phenomenon*, *semiotic bundle* and *semiotic mediation*
  - the theories of *variation*, the *cK¢ (conception, knowing, concept) model*
  - use of *discursive*, *embodied*, *ecocultural* and *material* perspectives
  - the framework of *instrumental genesis* is evident in research on the use of digital technologies

1.2 Summary of developments and trends in theories

- Researchers continue to find value and utility in the development and refining of theories that are specifically about the teaching and learning of geometry. The *theory of geometric work* is one recent development in this.
- The use of a wide variety of theoretical frames (such as *prototype theory*, *variation theory*, *the cK¢ (conception, knowing, concept) model*, various *semiotic*, *embodied*, *ecocultural* and *discursive* perspectives) illustrates the wide scope of geometry education research.
- Overall, during the past decade, there has been increased focus on *embodied* and *discursive* theories in research on the teaching and learning of geometry, with a concomitant research emphasis on *visuospatial reasoning*, on the use of *gestures* and *diagrams*, and on the use of *digital technologies*.

2. Advances in Understanding Visuospatial Reasoning

Led by Kay Owens and Nathalie Sinclair

1.1 Summary

- Visualising (mentally and physically) is well recognised as important in mathematics education but often not given sufficient emphasis in curriculum and teaching, perhaps because it is not easily assessed.
- Reasoning involves thinking about and making decisions based on visuospatial perception and understanding, both of which are influenced by prior knowledge and context of learning
- Visuospatial reasoning in geometry can be improved through experience from perception, through experience, to higher levels of reasoning (malleability) – evidence from psychology and education.

1.2 Educational Implications

- Locating: Young children use geometric features and landmarks to find their way around larger spaces – spatial as well as visual perception for decision-making. Cultural studies support cognitive science studies.
- Transformation: Mental rotation training improved algebraic manipulation, spatial abilities. Value of spatial versus object visualising.
- Value in spatially enriched education generally. Value in overcoming gender, culture, experience, and ability differences. Need for good visuospatial working memory in geometry.
- Origami, pop-up engineering; quality block play; practical activities; specific technology programs increased visuospatial reasoning.

1.3 Visuospatial reasoning in mathematics education research and sociocultural perspectives

- Drawing provided evidence of learning as well as being a mediating tool in learning.
- Complexity of reasoning about visuals; different impacts of different representations.
- Different activities create different imaginal, formational and trans-formational visuospatial reasoning. Processes of visual perception and perception-based knowledge influence learning.
- Impact of Western education may limit visuospatial reasoning of Indigenous, colonised groups; DGE can assist learning in developing communities as well as developed.

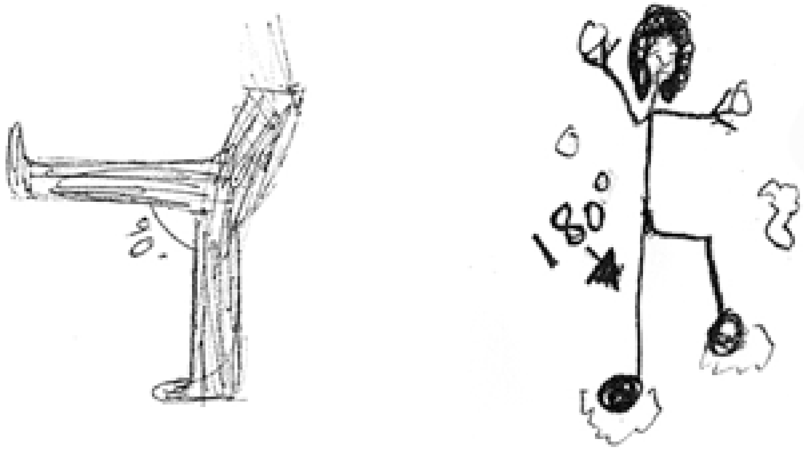
3. Gestures and diagrams

Led by Mariolina Bartolini Bussi and Kay Owens

THE ROLE OF SEMIOTIC PROCESSES AND ARTEFACTS

- Interplay between concrete artefacts and gestures.
- Historical roots of artefacts
- Important role of teacher

Embodiment: gestures and classroom applications

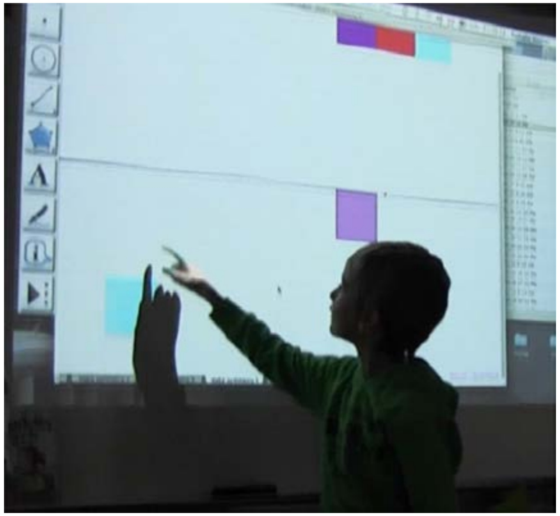


Fyhn (2008)
Climbing – leg angles

Healy & Fernandes (2011)
blind students - perimetre

Gestures and mathematical meanings

Bartolini Bussi &
Baccaglini-Frank (2015)
Bee-bot / square and
rectangle definitions



Ng & Sinclair (2015)
Symmetry – digital technology

4. The role of technology

Led by Ulrich Kortenkamp and Keith Jones

The Role of Technologies

- DGEs are being used widely, but more research is necessary to fully understand their role.
- Three important areas for research:
  - Introduction *and design* of new hard- and software
  - Theory and methodology for a better understanding of the role of existing and emerging technology
  - Empirical studies on the use of technology in teaching and learning

Trends for Digital Geometry Tools

- Geometry on the web faces various technological difficiculties; promising revival through JavaScript (JSXGraph, CindyJS, GeoGebra on GeoGebraTube)
- Interface design: How do we want to interact with (D)Geometry?
  - Mackrell 2011 – Design Decisions
  - K & Dohrmann 2010, Schimpf & Spannagel 2011 – UI
  - Jackiw & Sinclair 2009, Laborde & Laborde 2014 – 3 Dimensions of Transformation
- Mobile devices & Touch Technology
  - Concept of Dragging is being extended – New Modes of interaction, multitouch, collaboration

Specific Tools and Concepts

- Sliders
  - Different interaction – number oriented
- Socio-cultural aspects
  - Technology has social impact
- Spatial capabilities and 3D
  - Available, but currently restricted to 2d projection
- Task Design
  - Tasks change with availability of technology
- Assessment, Feedback, Learning Analytics
  - New approaches possible
- Teacher Education and PD
  - Challenging and important task for M.E.

The role of technology is just beginning to be understood, while, at the same time, it continues to evolve and rapidly change the world around us and in the classroom.

Students and teachers are using digital tools throughout the day, and it is necessary to better understand how they can be used effectively for teaching and learning.

5. Advances in the understanding of the teaching and learning of definitions

Led by Michael de Villiers and Mariolina Bartolini Bussi

5.1 Understanding the process of defining and the need for definitions

- Majority of studies concentrated on descriptive (*a posteriori*) defining, for example, defining a circle, triangles, quadrilaterals, and polyhedra after exploring their properties through DGE's, paper-folding, and/or pencil and paper construction
- But no studies specifically focused on students' deeper understanding of the need for axioms and definitions to avoid *infinite regress* and/or *circularity*

5.2 Understanding of Triangle and Quadrilateral Definitions

- The potential of DGE's, and some use of analogy also, in developing understanding for definitions were extensively explored in several studies with triangles and quadrilaterals, and appeared to have assisted students in developing more robust, dynamic concept images than the traditional prototypical, static images that tend to prevent inclusive definitions
- However, everyday language and prototypical conceptions remain an issue especially in regard to class-inclusion as well as students' understanding the constraints of a DGE figure
- No research on students' understanding of the need for definitions to avoid circularity, no use made of symmetry concepts in choice of definitions, and very little on engaging students in the process of the constructive (*a priori*) defining of new concepts.

6. THE TEACHING AND LEARNING OF THE PROVING PROCESS

Led by Allen Leung and Ulrich Kortenkamp

6.1 What is and what Constitutes a Mathematical Proof?

- Recent studies suggest that (geometrical) proof is socio-culturally bounded and is intimately related to the perceptual world.
  - What are alternative frameworks for "what is a geometrical proof is"
- A mathematical proof is closely tied with the corresponding conjecture or hypothesis, in particular, with how the conjecture or hypothesis came about.
- Due to social and technological advancements, empirical-based argument may play an important role in the formation of geometrical proof with respect to convincing or explaining.

6.2 DGE research on Conjecture Formation

- DGEs have been playing a vital epistemic role in studies that probed the process of generating geometrical conjectures.
- Through the lens of the theory of semiotic mediation (TSM), the conjecture production process is a semiotic process that involved a transformation from personal signs to mathematical signs.
- Feedback and mediation from technological tool serve as means for boundary crossing between the empirical and theoretical contexts in the proofing process. In particular, the DGE drag-mode instigates the complex interplays between inductive, abductive and deductive reasoning in the transition between empirical and theoretical proof perspectives.
- Studying and categorizing DGE dragging modalities/strategies have been a core focus attempting to conceptualize proof and explanation in DGE. Studies have been done to explore the role of DGE as an epistemic tool, in particular dragging, to open up a quasi-empirical dimension to the nature of proof, even indirect proof.

6.2 Geometry Proof in the Classroom

Pedagogies like tool-based task design, inquiry-based learning, mathematical discussion, problem modification, geometrical construction, even gesturing, have been introduced to improve the conjecture formation processes:

- Shift problem approach (empirical proof schemes, external conviction proof schemes, and deductive proof schemes)
- Teachers modified problems from a standard geometry textbook into investigation problems in DGE in their classes
- Students change through learning to construct mathematical proofs in an inquiry-based geometry class
- Flow-chart Proving
- Lens of cognitive unity addresses the tension between carrying out a geometrical construction and constructing the related proof
- Interplay among gestures, discourse and diagram in reasoning.

7. Moving beyond traditional Euclidean approaches

Led by Michael de Villiers

7.1 Focusing on 3D geometry

- Several studies indicated that students exhibited similar prototypical predispositions to 3D objects as they do with 2D objects
- Physically building, constructing and drawing 3D objects such as polyhedra, and/or exploring them dynamically with 3D DGE's appeared to develop better concept images and understanding of their properties
- Very few studies done on engaging students in extending interesting 2D results to 3D, for example, triangle concurrencies, Pythagoras, Varignon's or Viviani's theorem
- The use of analogy when moving from 2D to 3D and higher dimensions could be more extensively explored using analogous concepts for triangle, square, circles, perpendicular bisector, angle bisector, etc.

7.2 Non-Euclidean and other geometries

- Experimental studies on spherical and hyperbolical surfaces have used specific manipulatives such as spheres or DGE's to explore and prove results and properties of non-Euclidean objects, and in most cases, contrasting/comparing them with equivalents from Euclidean geometry
- Guven & Baki (2010) theorised van Hiele levels of understanding for spherical geometry similar to 2D, which appeared to be reasonably confirmed by a Guttman scalogram analysis, though future studies would be useful
- Two studies respectively looked at a Turtle geometry model of the hyperbolic surface, and at topological surfaces (the Mobius strip, the torus and the Klein bottle) using a DGE
- Little or no research on the teaching and learning of fractals in the past 10 years.

8. Concluding remarks

Looking to the future (1)

- Increased focus on embodied and discursive theories
  - For the future: How to provide teachers and students with more opportunities to engage in visuospatial reasoning?
  - How to adequately assess and value such reasoning?
- Strong presence of research involving the use of DGEs in a wide range of contexts
  - For the future: more research needed on task design and on teacher practice
  - Possible relevance to recent push on coding/computational thinking

Looking to the future (2)

- Broadening of the traditional scope of geometry, both in terms of cultural perspectives and also in moving beyond the Euclidean development
- Continued research on how to promote more geometry and spatial reasoning in the curriculum, as number and algebra often overemphasised.
- Increased attention to contemporary mathematics, which is currently experiencing a major programme of geometrisation (Dieudonné, Zalamea)

Thank you!

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