

Thematic Afternoon at ICME 13: „Legacy of Felix Klein”

Organizing committee:

Hans-Georg Weigand (University of Wuerzburg, Germany)¹ – Chair

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Introduction:

The Thematic Afternoon “The Legacy of Felix Klein” will give an overview about the ideas of Felix Klein, it will highlight some developments in university teaching and school mathematics related to Felix Klein in the last century, and it will especially be asked for the meaning, the importance and the legacy of Kleins’ ideas nowadays and in the future in an international, worldwide context.

The Thematic Afternoon at ICME 13, Wednesday, 27 July 2016, is divided into two parts. The first one-hour timeslot will be a panel or a plenary. The second two-hour timeslot consists of three “strands”.

- Strand A: Functional thinking
- Strand B: Intuitive thinking and visualization
- Strand C: Elementary Mathematics from a higher standpoint

The whole Thematic Afternoon of course will include international aspects. To be attractive for the audience there will be different forms of presentations and some diversions planned during the two hours: There will be historic texts and hands-ons, pictures (especially of geometric models from Felix Klein), films, video-clips or short movies, authentic view into classrooms, real models of geometric surfaces and reconstructions of geometric models with a 3D printer.

Timeslot 1: Panel (3pm-4pm)

- What is and what could be the legacy of Felix Klein? **Hans-Georg Weigand** (Würzburg, GER)

15 min

Felix Klein always emphasized the great importance of teaching at the university, and he strongly promoted the modernization of mathematics in the classroom. The three books “Elementary Mathematics from a higher (advanced) standpoint” from the beginning of the last century gave and still give a model for university lectures especially for student teachers. The “Merano Resolution” (1905), essentially initiated and influenced by Felix Klein, pleaded for an orientation of mathematics education at the concept of function, an increased emphasis on spatial geometry and an introduction of calculus in high schools. Moreover, his (didactic) principles concerning the greater meaning of visualization, applications and connections are worth to be discussed in the frame of contemporary research.

- Biographical notes: **Renate Tobies**⁶ (Jena, GER):

15 min

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⁶ <http://www.mathematik.uni-kl.de/~tobies/>

Felix Klein and his comprehensive program to promote mathematics, its applications, and mathematical instruction

Having been a full professor at the University of Erlangen, the Technical University in Munich, and the University of Leipzig, Klein joined the University of Göttingen in 1886. He had gained international recognition with his significant achievements in the fields of geometry, algebra, and the theory of functions. On this basis, he was able to create a center for mathematical and scientific research in Göttingen.

This biographical note will demonstrate that Felix Klein was far ahead of his time in supporting all avenues of mathematics, its applications, and mathematical pedagogy. It will be shown that the establishment of new lectures, professorships, institutes, and curricula went hand in hand with the creation of new examination requirements for prospective secondary school teachers. In Germany, all mathematical instruction at universities was aimed at training future teachers.

- Introduction into or information about the planned content of the different strands:
 - Strand A: Bill McCallum: Functional thinking **10 min**
 - Strand B: Michael Neubrand: Intuitive thinking and visualization **10 min**
 - Strand C: Marta Menghini & Gert Schubring: Elementary Mathematics from a higher standpoint **10 min**

Coffee Break (4.00pm-4.30pm)

Timeslot 2: Strands (4.30pm-6.30pm)

Strand A: Functional Thinking

Organizer: Bill McCallum (USA)

Klein's chapter on analysis considered functions from many points of view, from functions defined by power series and Fourier series, to functions defined (intuitively) by their graphs, to functions defined abstractly as mappings from one set to another. He made an important distinction between functions arising out of applications of mathematics and functions as abstractions in their own right. This distinction reverberates in mathematics education today. Strand A will consider the development of the concept of function and its role in K-12 mathematics education from Klein's time to the present day. It will include a presentation of Klein's original thinking in this area, a tracing of the history of conceptual development of functional thinking, a discussion of the influence of the concept of function on curriculum, and ways in which the concept of function is presented in modern curricula K-12.

Speakers who have agreed to present include

- **Hyman Bass** (Michigan, USA)
- **Katja Krüger** (Paderborn, GER), and
- **Pat Thompson** (Arizona, USA).

Each hour will consist of an introductory 20 minute presentation, followed by a 20 minute panel discussion, followed 20 minutes of questions from the audience.

Functional Thinking – about the history of a didactical principle

Katja Krüger

The demand for education to functional thinking (“Funktionales Denken”) was explicitly penned for the first time in the so called Meran curricula (“Meraner Lehrpläne”) in 1905 and has become a widely accepted catch-phrase in Germany. Establishing the habit of functional thinking in higher maths education was one of the major goals of the Prussian reform-movement at the beginning of the 20th century. It had a great impact on the whole German school system. So what was it, the curriculum reformers around Felix Klein meant by this notion? This presentation illustrates with examples taken from contemporary schoolbooks and publications that functional thinking didn’t mean teaching “the” concept of function as we understand it nowadays. Actually, it focusses on a certain kinematic habit of mind that can be described by investigating changeability and movement.

The K-12 Number Line: Is it built, or occupied?

Hyman Bass

At the end of high school Anne can meaningfully hear, “Let $f(x)$ be a function of a real variable x .” The home of this x is the real number continuum \mathbf{R} . How did this \mathbf{R} , with its rich algebraic and geometric structure, make itself progressively known to the first grade Anne, presumed to know little more than simple cardinal counting? There are two possible narratives meant to accomplish this.

The **construction narrative** starts with counting numbers, and gradually builds \mathbf{R} by accretion of new numbers (negative integers, fractions) and eventually a “hole filling” completion (from \mathbf{Q} to \mathbf{R}) whose need and nature are often left tacit. A further difficulty with this narrative is that fractions appear, conceptually, notationally, and computationally, as a whole new number planet, whose integration into the eventual number continuum, that it cohabits with integers and rational numbers, is complex and non intuitive. The construction narrative, characteristic of much of the U.S. curriculum, risks not achieving the final synthesis, an internalized understanding of the real number continuum

The **occupation narrative** (promoted by V. Davydov & colleagues), in contrast, treats the geometric line as present from the beginning; it is the natural environment for linear measurement. This confers on the geometric line (imagine a thin string, flexible but inelastic) a primordial cognitive status on the same footing as cardinal counting. Measurement is at first of *quantities* (continuous as well as discrete). *Numbers* then arise as *ratios* of two quantities, one taken as a unit against which to measure the other. In this way, once an (oriented) interval on the line is chosen as unit, all intervals acquire, at least conceptually, a numerical measure (or ratio), and so, the continuum of all real numbers is, at least conceptually, present from the beginning. The progression in the construction narrative above, is now replaced by the progressive naming of more and more of these numbers, as we locate where they take up residence on the (pre-existing) line.

USA and South Korean teachers' meanings for function and function notation as a potential source of differences in students' learning

Pat Thompson

The US National Council of Teachers of Mathematics, in its Principles to Actions (NCTM, 2014), said "Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments." But, a teacher will not consciously support the building of a shared understanding that the teacher does not share. I will report a study of ways that 253 USA and 368 South Korean teachers understand the ideas of function and function notation, and will provide researchable hypotheses about their potential effects on student learning.

The first hour

The varieties of functional thinking

As Klein indicated in *Elementary Mathematics from an Advanced Standpoint*, the notion of a function has many aspects: analytic, geometric, and abstract. Each aspect has both a historical development and a pedagogical profile. We will consider both of these, and also the extent to which they correspond.

The second hour

Functions in the modern curriculum

In this hour we will explore concrete examples of functional thinking in the modern curriculum. We will consider examples where the abstract notion has been ignored and examples where it has been over-emphasized. We will consider curricula where functions are introduced early and those where they are introduced late. We will pose the question of whether Klein's vision for the K-12 has been realized, and whether it is still relevant.

Strand B: Intuitive thinking and visualization

Organizer Michael Neubrand (GER)

The first hour: 16.30 – 17.30

Klein's ideas on promoting visual thinking

1 Martin Mattheis (Mainz, Germany):

Aspects of "Anschauung" in the work of Felix Klein

The so called „Klein-Reform-Movement“ about the teaching of mathematics in the secondary schools in Prussia about 1900 was marked by two theses: "Anschauung" (perception), and "funktionales Denken" (thinking based on functionality). The lecture will present a few examples illustrating the role of "Anschauung" in the work of Felix Klein.

2 Stefan Halverscheid (Göttingen, Germany) and Oliver Labs (Potsdam, Germany):

Felix Klein's mathematical heritage seen through 3D-models and other tools of visualisation

Building-up intuition with the help of visualisations was a major goal of Felix Klein's vision of how mathematics should be done and taught. In his famous Erlangen speech (Klein, 1872) he said: „Instead of developing a proper feel for mathematical operations or promoting a lively, intuitive grasp of geometry, the class-time is spent learning mindless

formalities or practicing pretty trivialities that exhibit no underlying principle".
Four underlying principles which played a major role in Klein's mathematical work are considered more closely: the interplay between abstraction and visualisation, discovering the nature of objects with the help of small changes, functional thinking and, finally, the characterisation of geometries in the way it is used till today.
The principles are discussed with respect to their particular potential for the roles of intuition and of visualisations for the teaching and learning of mathematics - both then and now. Special emphasis is given to illustrations and examples: Drawings, models, films.

The second hour: 17.35 – 18.30

The impact of Klein's ideas on visualization and intuition into modern teaching mathematics

3 Flavia Mammana (Catania, Italy):

The modernity of the Meraner Lehrplan for teaching geometry

In 1905 Felix Klein proposed a mathematics curriculum, the "Meraner Lehrplan". In this document he proposed an approach to geometry via intuitive geometry, seen as ability to see in space, in order to provide, on the one hand, elements for interpreting the real world, on the other hand to develop logical skills.

Klein's ideas still hold today: an intuitive approach to geometry is facilitated by the use of information technology. We intend to present some activities in grades 10-11 related to the approach of space geometry, which are based on both, the analogy among figures and the use of dynamic geometry software.

4 Chris Rasmussen (San Diego, USA):

Visualization and Intuition in Linear Algebra

Compatible with Klein's view that mathematical ideas naturally evolve from a naïve state to refined state, this contribution presents an innovative instructional sequence for an introductory linear algebra course that supports students' reinvention of the concepts of span, linear dependence, and linear independence. Referred to as the Magic Carpet Ride sequence, the problems begin with an imaginary scenario that allows students to build rich imagery and formal definitions. The approach begins by focusing on vectors, their algebraic and geometric representations in 2 and 3 dimensions, and their properties as sets. Samples of student work are provided to illustrate the variety of student solutions and how these solutions lead to the creation of formal definitions

5 Ysette Weiss-Pidstrygach (Mainz, Germany):

Grasping Mathematics: Using Treutlein's Classroom Models in Teacher Education

The spirit of Felix Klein in using models for teaching was carried on from university to school by Peter Treutlein, already before the first world war. Today, his models offer a rich variety of possibilities to relate historical, mathematical and pedagogical aspects in mathematics teacher education. With such models Klein and Treutlein explicitly attempted to transform visual perception into logical concepts and showed thereby a modern view on concept development.

... and at the end ...

6 Sebastian Kitz (Wuppertal, Germany):

Mathematical animated films as teaching material

Felix Klein's idea to emphasize the dynamical view of geometrical objects resulted in different kinds of teaching material. Especially the 'Mathematical animated films' of Ludwig Münch give a good impression of the progress achieved at that time. They were

produced about 1910 and can be seen as a precursor of modern dynamic geometry environments.

Strand C

Elementary Mathematics from a Higher Standpoint – Conception, realization, and impact on teacher education

Organizers: Marta Menghini (IT), Gert Schubring (GER/BRA)

Strand C will be concerned with the three volumes on “Elementary mathematics from a higher standpoint” (which will be available in their new complete English edition by ICME 13): their conception, realization, and impact on teacher education. These lecture notes from the early 20th century were a seminal contribution by Klein to mathematics teacher education, presenting for the first time a methodological orientation for teacher education, not just a content oriented course. It constituted a model for many later approaches. Strand C intends to assess the importance of this conception of mathematics teacher education, to discuss its international impact and to reflect about its impact for present-day teacher education. This regards in particular volume III, on the relation between applied and pure mathematics – a subject of utmost concern for Klein -, which has hitherto not been known internationally.

16.30 – 17.30: The content of three volumes on “Elementary mathematics from a higher standpoint”

Gert Schubring

Klein’s conception of ‘Elementary Mathematics from a Higher Standpoint’

ABSTRACT: The notion of school mathematics uses to be discussed historically between the two extreme positions of transposition didactique (Chevallard) and histoire des disciplines scolaires (Chervel). Felix Klein’s innovative establishment of „elementary“ is analysed here: not in the common-day meaning of „simple“, but as the result of the process of elementarisation of complex developments in mathematics, effecting a restructuration of mathematics from new conceived elements. Klein does not prescribe schools to adopt the latest developments in science and rather allows them to make proper choices according to criteria of the school system, yielding a certain „hysteresis“ behind the recent, not yet elementarised state.

Marta Menghini

The importance of Volume III in Klein’s series of lecture notes

ABSTRACT: In the third volume of EMFHS, “Precision and approximation mathematics” Klein uses circular inversion to introduce point sets with particular properties. This example, that will be shown using Geogebra, allows to illustrate some features of Klein’s text: starting from an intuitive and sometimes practical approach, Klein develops abstract concepts working in rich “mathematical environments”, which form the core of an interesting high school mathematics teacher education. Another example concerns the “continuous” transformation of curves, evidencing the invariant properties.

Henrike Allmendinger

Examples of Klein's practice

ABSTRACT: In the first volume of EMFHS, "Arithmetic, Algebra and Analysis", Klein closely adheres to several principles, which contribute to the understanding of Klein's higher standpoint to a large amount. In addition, Klein conveys not only to a mathematical, but also to a historical and didactical perspective, that broaden this standpoint. This versatile approach to the mathematical content will be illustrated by taking a closer look at original text snippets from the chapter on Logarithmic and Exponential Functions (Beamertex-presentation as a pdf-file)

17.30 – 17.45 Jeremy Kilpatrick

Comments on the previous part and introduction to the second part

17.45 – 18.30: The impact on teacher education

Frédéric Gourdeau

Disciplinary mathematics and school mathematics for teacher education today

ABSTRACT: The emphasis placed upon content in the mathematical education of teachers is still being questioned today, just as Klein questioned it in his seminal work. Recently, the notion of "Mathematical Habits of Mind" has been at the forefront of some discussions. This is exemplified by recent proposals for curriculum reform in Canada. In my presentation, the interplay between these facets of mathematics education will be discussed in the context of mathematics for teacher education.

Masami Isoda

Impact of Klein's work for teacher education today: Asia

ABSTRACT: The Japanese Secondary School Textbook of 1943 reveals pertinent influences of Klein and his collaborator Horst von Sanden. The textbook was also used in some parts of East Asia. The basic principle of the textbook is mathematisation and this provided after WW II the roots of the curriculum principle 'Extension and Integration' in Japan.

Katalin Gosztonyi

Impact of Klein's work for teacher education today: East Europe

ABSTRACT: In this presentation, I will look for possible traces of Felix Klein's influence on Hungarian mathematics education, from his student and colleague M. Beke, through mathematicians at the mid-20th century as L. Kalmár and R. Péter, until the reform of mathematics education led by T. Varga in the 1970's. I will describe common aspects of their conception about mathematics like the emphasis on intuition, visuality, organic development or cultural aspects of mathematics (Power Point presentation; using of Klein's and Beke's writings).

